

Characteristic functions under series and parallel connection of quantum graphs

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Abstract. For a graph consisting of parallel connected subgraphs we express the characteristic function of the boundary value problem with generalized Neumann conditions at both joining points via characteristic functions of different boundary problems on the subgraphs.

Keywords: Dirichlet boundary condition, Neumann boundary condition, Kirchhoff condition, continuity condition, subgraph, connectivity, boundary value problem, spectral parameter.

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1. Introduction

We consider boundary value problems generated by the Sturm-Liouville equation on a connected compact metric graph with continuity and Kirchhoff's conditions at interior vertices and Robin or Dirichlet conditions at pendant vertices (i.e. vertices of degree 1). The potentials in the Sturm-Liouville equations and constants in the conditions are assumed to be real, and such that the corresponding operator is self-adjoint.

To describe the spectrum of a boundary value problems on metric graphs it is possible to use characteristic functions or spectral determinants.

It was proved in [2] that the characteristic function $\Phi_N(\lambda)$ of the boundary value problem on a graph G which consists of two subgraphs G_1 and G_2 with the generalized Neumann (i.e. continuity + Kirchhoff's) boundary condition at the only cut-vertex (see [11], p.54 for a definition) \mathbf{v} of G satisfies

$$\Phi_N(\lambda) = \Phi_N^1(\lambda)\Phi_D^2(\lambda) + \Phi_D^1(\lambda)\Phi_N^2(\lambda), \quad (1.1)$$

where $\Phi_N^j(\lambda)$ ($j = 1, 2$) are the characteristic functions of boundary value problems on subgraphs G_1 and G_2 with Neumann conditions at \mathbf{v} , and $\Phi_D^j(\lambda)$ are the characteristic functions of boundary value problems on G_1 and G_2 with Dirichlet condition at \mathbf{v} .

Formula (1.1) was proved in [2] for trees but the proof is quite the same for any separable graph. Earlier this formula was obtained for the so-called spectral determinants [4]. In case of Neumann conditions at pendant vertices and under the condition of no loops and multiple edges the spectral determinant is nothing but our characteristic function. Also it should be mentioned that in [4] a definition of spectral determinant introduced in [5] was used which is not quite accurate. The solutions $\psi_{\alpha\beta}$ and $\psi_{\beta\alpha}$ defined by (1)–(3) in [5] not always exist, e.g. in case of $\gamma = \alpha = 0$, $\beta = \pi$ (in terms of [5]) these solutions are absent. Therefore we use the characteristic functions language in the present paper to be mathematically rigorous. Formula (1.1) for the graph P_2 (a path with two edges) has been used for solving the so-called inverse three spectra problem [6] and the Hochstadt-Lieberman problem [7] and for describing spectral problems generated by equations of Stieltjes strings on trees in [8], [9], [10].

A graph of connectivity 1 (see [1], p.75 or [11], p.72 for a definition), i.e. a separable graph can be considered as a series connection of its subgraphs. In the present paper we consider a parallel connection of subgraphs into a graph of connectivity 2 aiming to deduce an analogue of (1.1) for parallel connection of subgraphs.

2. Characteristic functions

Characteristic functions of boundary value problems on graphs are natural generalizations of characteristic functions of boundary value problems on an interval. For an interval we use the following four characteristic functions. If $s(\lambda, x)$ is the solution

of the Sturm-Liouville equation with a real valued potential $q(x) \in W_2^2(0, l)$:

$$-y'' + q(x)y = \lambda^2 y, \quad x \in (0, l), \quad (2.1)$$

which satisfies the conditions $s(\lambda, 0) = s'(\lambda, 0) - 1 = 0$ and $c(\lambda, x)$ is the solution which satisfies $c(\lambda, 0) - 1 = c'(\lambda, 0) = 0$, then we call $\Phi_{DD}(\lambda) \stackrel{\text{def}}{=} s(\lambda, l)$ the *Dirichlet-Dirichlet* characteristic function because the set zeros of $\Phi_{DD}(\lambda)$ coincides with the spectrum of the problem generated by (2.1) and the boundary conditions

$$y(0) = y(l) = 0.$$

We call $\Phi_{DN}(\lambda) \stackrel{\text{def}}{=} s'(\lambda, l)$ the *Dirichlet-Neumann* characteristic function because the set of zeros of $\Phi_{DN}(\lambda)$ coincides with the spectrum of the problem generated by (2.1) and the boundary conditions

$$y(0) = y'(l) = 0,$$

we call $\Phi_{ND}(\lambda) \stackrel{\text{def}}{=} c(\lambda, l)$ the *Neumann-Dirichlet* characteristic function because the set of zeros of $\Phi_{ND}(\lambda)$ coincides with the spectrum of the problem generated by (2.1) and the boundary conditions

$$y'(0) = y(l) = 0,$$

we call $\Phi_{NN}(\lambda) \stackrel{\text{def}}{=} c'(\lambda, l)$ the *Neumann-Neumann* characteristic function because the set of zeros of $\Phi_{NN}(\lambda)$ coincides with the spectrum of the problem generated by (2.1) and the boundary conditions

$$y'(0) = y'(l) = 0.$$

Now let us consider boundary value problems on a compact connected graph. Let G be a metric graph with g edges. We denote by v_j the vertices of G , by $d(v_j)$ their degrees, by e_j the edges of G and by l_j their lengths. An arbitrary vertex \mathbf{v} is chosen as the root. Since as it will be clear below our results do not depend on the orientation of the edges, we fix an arbitrary orientation but for the sake of convinience we assume that the root \mathbf{v} has only outgoing incident edges. Local coordinates for edges identify the edge e_j with the interval $[0, l_j]$ so that the local coordinate increases in the direction of the edge. This means that the root \mathbf{v} has the local coordinate 0 on each incident edge. All the other interior vertices v may have outgoing edges, with local coordinates 0, and incoming edges e_j with local coordinates l_j . Functions y_j on the edges are subject to the scalar Sturm- Liouville equations:

$$y_j'' + q_j(x_j)y_j = \lambda^2 y_j, \quad (2.2)$$

where q_j is a real-valued function which belongs to $L_2[0, l_j]$. For an edge e_j incident to a pendant vertex which is not the root we impose self-adjoint boundary conditions

$$y_j'(0) + \beta_j y_j(0) = 0, \quad (2.3)$$

or

$$y_j'(l_j) + \beta_j y_j(l_j) = 0, \quad (2.4)$$

where $\beta_j \in R \cup \{\infty\}$. The case $\beta_j = \infty$ corresponds to Dirichlet boundary condition $y_j(0) = 0$ or $y_j(l_j) = 0$.

For each interior vertex which is not the root with incoming edges e_j and outgoing edges e_k the continuity conditions are

$$y_j(l_j) = y_k(0), \quad (2.5)$$

for all j and k and Kirchhoff's condition is

$$\sum_k y'_k(0) = \sum_j y'_j(l_j). \quad (2.6)$$

At the root \mathbf{v} , we impose the continuity conditions

$$y_j(0) = y_k(0) \quad (2.7)$$

for all edges e_j and e_k outgoing from \mathbf{v} and Kirchhoff's condition

$$\sum_k y'_k(0) = 0 \quad (2.8)$$

where the sum in the left-hand side is taken over all edges outgoing from \mathbf{v} .

We will call the pairs of conditions (2.5)-(2.6) and (2.7)-(2.8) *generalized Neumann conditions* for an interior vertex. It is clear that being imposed at a pendant vertex these conditions are reduced to the usual Neumann condition (condition (2.3) or (2.4) with $\beta_j = 0$).

Let us denote by $s_j(\lambda, x_j)$ the solution of the Sturm-Liouville equation (2.2) on an edge e_j which satisfies the conditions $s_j(\lambda, 0) = s'_j(\lambda, 0) - 1 = 0$ and by $c_j(\lambda, x_j)$ the solution which satisfies the conditions $c_j(\lambda, 0) - 1 = c'_j(\lambda, 0) = 0$. Then the *characteristic function*, i.e. an entire function whose zeros coincide with the spectrum of the problem can be expressed via $s_j(\lambda, l_j)$, $s'_j(\lambda, l_j)$, $c_j(\lambda, l_j)$ and $c'_j(\lambda, l_j)$. To do it we introduce the following system of vector functions $\psi_j(\lambda, \vec{x}) = \{0, 0, \dots, c_j(\lambda, x_j), \dots, 0\}^T$ and $\psi_{j+n}(\lambda, \vec{x}) = \{0, 0, \dots, s_j(\lambda, x_j), \dots, 0\}^T$ for $j = 1, 2, \dots, g$, where g is the number of edges in the graph, $\vec{x} = \{x_1, x_2, \dots, x_g\}^T$. As in [3], we denote by L_j ($j = 1, 2, \dots, 2g$) the linear functionals generated by (2.2)–(2.8). Then $\Phi(\lambda) = (L_j(\psi_k(\lambda, \vec{l})))_{j,k}^{2g}$ where $\vec{l} = \{l_1, l_2, \dots, l_g\}^T$ is the characteristic matrix which represents the system of linear equations describing the continuity and Kirchhoff conditions for the interior vertices. Then we call

$$\phi_N(\lambda) := \det(\Phi(\lambda))$$

the *characteristic function* of problem (2.2)–(2.8). We are interested also in the problem generated by the same equations and all boundary and matching conditions the same, but with the condition

$$y_j(0) = 0 \quad (2.9)$$

instead of (2.8) at \mathbf{v} . We denote the characteristic function of problem (2.2)–(2.7), (2.9), by $\Phi_D(\lambda)$. We call (2.7), (2.9) the *generalized Dirichlet condition*. In case when \mathbf{v} is a pendant vertex, conditions (2.7), (2.9) are reduced to the usual Dirichlet boundary condition. Let us assume that the graph G is separable, i.e of connectivity 1 and

let the root \mathbf{v} be the cut-vertex. We divide our graph G into two subgraphs G_1 and G_2 having the only common vertex \mathbf{v} . Denote by $\Phi_N^j(\lambda)$ ($j = 1, 2$) the characteristic function corresponding to subgraph G_j with Neumann condition at \mathbf{v} and by $\Phi_D^j(\lambda)$ the characteristic function corresponding to subgraph G_j with Dirichlet condition at \mathbf{v} .

Formula (1.1) describes series connection of subgraphs. In order to describe the characteristic functions for series and parallel connected subgraphs we need to generalize the notions of Φ_{DD} , Φ_{DN} , Φ_{ND} and Φ_{NN} introduced in the beginning of this Section for boundary value problems on an interval.

Let the vertices \mathbf{v}_{in} and \mathbf{v}_{out} be pendant vertices and let the edge e_1 be outgoing away from \mathbf{v}_{in} and the edge e_g be incoming to \mathbf{v}_{out} . Denote by $w = \mathbf{v}_{out}$ the second vertex incident with e_g . Let w have no incoming edges and denote the outgoing from w edges by $e_{g-d(w)+1}, \dots, e_g$.

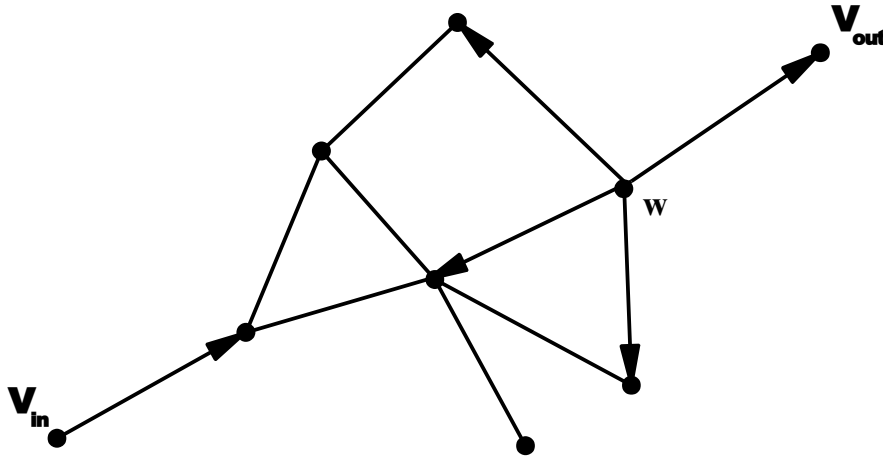


Fig.1

Let us consider the Dirichlet-Dirichlet problem (2.2)–(2.7), (2.9) which in our terms is as follows:

$$y_1(0) = 0 \tag{2.10}$$

at \mathbf{v}_{in} ,

$$y'_j(0) + \beta_j y_j(0) = 0, \tag{2.11}$$

or

$$y'_j(l_j) + \beta_j y_j(l_j) = 0, \tag{2.12}$$

at each pendant vertex except of \mathbf{v}_{in} and \mathbf{v}_{out} . For each interior vertex which is not w with incoming edges e_j and outgoing edge e_k the continuity conditions are

$$y_j(l_j) = y_k(0), \tag{2.13}$$

and Kirchhoff's condition is

$$\sum_k y'_k(0) = \sum_j y'_j(l_j). \quad (2.14)$$

At w we have

$$y_{g-d(w)+1}(0) = y_{g-d(w)+2}(0) = \dots = y_{g-1}(0) = y_g(0), \quad (2.15)$$

and

$$y'_g(0) = - \sum_{k=g-d(w)+1}^{g-1} y'_k(0) \quad (2.16)$$

and

$$y_g(l_g) = 0 \quad (2.17)$$

at \mathbf{v}_{out} .

We substitute

$$y_i = B_i c_i(\lambda, x_i) + A_i s_i(\lambda, x_i)$$

into (2.10)–(2.17) and obtain

$$B_1 = 0, \quad (2.18)$$

$$A_j + \beta_j B_j = 0, \quad (2.19)$$

$$B_j c'_j(\lambda, l_j) + A_j s'_j(\lambda, l_j) + \beta_j (B_j c_j(\lambda, l_j) + A_j s_j(\lambda, l_j)) = 0. \quad (2.20)$$

We can also write the continuity and Kirchhoff's condition at \tilde{v} as

$$B_{g-d(w)+1} = B_{g-d(w)+2} = \dots = B_{g-1} \quad (2.21)$$

$$B_{g-1} = B_g, \quad (2.22)$$

$$A_g = - \sum_{k=g-d(w)+1}^{g-1} A_k, \quad (2.23)$$

$$B_g c_g(\lambda, l_g) + A_g s_g(\lambda, l_g) = 0. \quad (2.24)$$

These equations can be written in a matrix form:

$$\begin{pmatrix} 1 & 0 & \dots & \dots & \dots & 0 & 0 & 0 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \times & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \times & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \times & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \times & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 & -1 & 0 & 0 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \times & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \times & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \times & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \times & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & 0 & 0 & 0 & \dots & 1 & 1 & 1 \\ 0 & 0 & \dots & \dots & 0 & c_g & 0 & \dots & \dots & \dots & 0 & s_g \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ \vdots \\ B_{g-1} \\ B_g \\ A_1 \\ A_2 \\ \vdots \\ \vdots \\ A_{g-1} \\ A_g \end{pmatrix} = 0. \quad (2.25)$$

Here $s_j = s_j(\lambda, l_j)$, $s'_j = s'_j(\lambda, l_j)$, $c_j = c_j(\lambda, l_j)$, $c'_j = c'_j(\lambda, l_j)$.

The determinant of the matrix in (2.24) is $\Phi_{DD}(\lambda)$.

Let us delete equations (2.18), (2.22), (2.23) and (2.24) from (2.25) and set $B_1 = B_g = A_g = 0$ and $A_1 = 1$. Then we obtain a nonhomogeneous $(g-4) \times (g-4)$ system of linear algebraic equation with respect to unknowns $B_2, B_3, \dots, B_{g-1}, A_2, A_3, \dots, A_{g-1}$:

$$\begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} B_2 \\ \cdot \\ \cdot \\ B_{g-1} \\ A_2 \\ \cdot \\ \cdot \\ A_{g-1} \end{pmatrix} = - \begin{pmatrix} \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \end{pmatrix}, \quad (2.26)$$

Denote the determinant of the matrix of the obtained system by $\Delta(\lambda)$. Solving this system we find

$$B_{g-1}^D = \frac{\Delta_{B_{g-1}}^D(\lambda)}{\Delta(\lambda)}, \quad A_{g-k}^D = \frac{\Delta_{A_{g-k}}^D(\lambda)}{\Delta(\lambda)} \quad k = 1, 2, \dots, d(w) - 1, \quad (2.27)$$

where $\Delta_{B_{g-1}}^D$ and $\Delta_{A_k}^D$ are the corresponding cofactors:

$$\Delta_{B_{g-1}}^D(\lambda) = \det \begin{vmatrix} \dots & \dots & \dots & \times & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \times & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \times & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \times & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \times & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \times & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \times & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \times & \dots & \dots & \dots & \dots \end{vmatrix},$$

$$\Delta_{A_{g-1}}^D(\lambda) = \det \begin{vmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \times \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \times \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \times \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \times \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \times \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \times \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \times \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \times \end{vmatrix}, \quad \Delta_{A_{g-2}}^D(\lambda) = \det \begin{vmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \times & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \times & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \times & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \times & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \times & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \times & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \times & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \times & \dots \end{vmatrix}$$

and so on up to $\Delta_{A_{g-d(w)+1}}^D$.

We use upper index D to underline that the formulae correspond to the case of Dirichlet boundary condition at \mathbf{v}_{in} . Notice that Δ does not depend on the condition

at \mathbf{v}_{in} while it is the determinant of the matrix obtained from the matrix of (2.27) by deleting among others the first column and the $(g+1)$ -th column.

On the other hand, it is easy to notice that these cofactors appear in the expansion of the determinant

$$\Phi_{DD}(\lambda) = c_g(\lambda, l_g) \tilde{\Delta}_{B_{g-1}}^D(\lambda) + s_g(\lambda, l_g) \sum_{k=g-d(w)+1}^{g-1} \tilde{\Delta}_{A_k}^D(\lambda) \quad (2.28)$$

because it is easy to see that $\tilde{\Delta}_{B_{g-1}}^D(\lambda) = \Delta_{B_{g-1}}^D(\lambda)$ and $\tilde{\Delta}_{A_k}^D(\lambda) = \Delta_{A_k}^D(\lambda)$.

In the same way we arrive at

$$\Phi_{DN}(\lambda) = c'_g(\lambda, l_g) \Delta_{B_{g-1}}^D(\lambda) + s'_g(\lambda, l_g) \sum_{k=g-d(w)+1}^{g-1} \Delta_{A_k}^D(\lambda). \quad (2.29)$$

Now we consider the same problem but with Neumann condition $y'_1(0) = 0$ at \mathbf{v}_{in} . Then we obtain $A_1 = 0$ instead of (2.18) and

$$\begin{pmatrix} 0 & 0 & \dots & \dots & \dots & 0 & 1 & 0 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 & -1 & 0 & 0 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 1 & -1 \\ 0 & 0 & \dots & \dots & 0 & c_g & 0 & \dots & \dots & \dots & 0 & s_g \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ \cdot \\ \cdot \\ B_{g-1} \\ B_g \\ A_1 \\ A_2 \\ \cdot \\ \cdot \\ A_{g-1} \\ A_g \end{pmatrix} = 0$$

instead of (2.25). In the same way as (2.27) we obtain

$$B_{g-1}^N = \frac{\Delta_{B_{g-1}}^N(\lambda)}{\Delta(\lambda)}, \quad A_{g-k}^N = \frac{\Delta_{A_{g-k}}^N(\lambda)}{\Delta(\lambda)} \quad k = 1, 2, \dots, d(w) - 1, \quad (2.30)$$

where $\Delta_{B_{g-1}}^N$ and $\Delta_{A_k}^N$ are the corresponding cofactors. Upper index N we use to underline that the formulae correspond to the case of Neumann boundary condition at \mathbf{v}_{in} . Also we have

$$\Phi_{ND}(\lambda) = c_g(\lambda, l_g) \Delta_{B_{g-1}}^N(\lambda) + s_g(\lambda, l_g) \sum_{k=g-d(w)+1}^{g-1} \Delta_{A_k}^N(\lambda),$$

$$\Phi_{NN}(\lambda) = c'_g(\lambda, l_g) \Delta_{B_{g-1}}^N(\lambda) + s'_g(\lambda, l_g) \sum_{k=g-d(w)+1}^{g-1} \Delta_{A_k}^N(\lambda).$$

3. Series connection

Let us consider two connected graphs G^j ($j = 1, 2$). We choose two vertices \mathbf{v}_{in}^j and \mathbf{v}_{out}^j in each of them as the entrance and exit vertices. In this Section we investigate series connection of these graphs. We denote by Φ_{NN}^j the characteristic function of the boundary value problem on the graph G_j with Neumann boundary conditions at \mathbf{v}_{in}^j and \mathbf{v}_{out}^j . At all the other interior vertices generalized Neumann conditions (continuity and Kirchhoff conditions) are imposed while any self-adjoint conditions of the form (2.3) or (2.4) are imposed at pendant vertices. In the same way, we denote by Φ_{ND}^j the characteristic function of the boundary value problem on the graph G_j with Neumann boundary condition at \mathbf{v}_{in}^j and Dirichlet boundary condition at \mathbf{v}_{out}^j , by Φ_{DN}^j the characteristic function of the boundary value problem on the graph G_j with Dirichlet boundary condition at \mathbf{v}_{in}^j and Neumann boundary condition at \mathbf{v}_{out}^j , by Φ_{DD}^j the characteristic function of the boundary value problem on the graph G_j with Dirichlet boundary condition at \mathbf{v}_{in}^j and at \mathbf{v}_{out}^j .

If we connect \mathbf{v}_{out}^1 with \mathbf{v}_{in}^2 then we obtain a new graph $G = G_1 \cup G_2$ with a cut-vertex $\mathbf{v}_{out}^1 = \mathbf{v}_{in}^2$. Let us denote by $\Phi_{NN}(\lambda)$, $\Phi_{ND}(\lambda)$, $\Phi_{DN}(\lambda)$ and $\Phi_{DD}(\lambda)$ the characteristic functions of the problems on G with Neumann conditions at \mathbf{v}_{in}^1 and \mathbf{v}_{out}^2 , with Neumann conditions at \mathbf{v}_{in}^1 and Dirichlet at \mathbf{v}_{out}^2 , with Dirichlet condition at \mathbf{v}_{in}^1 and Neumann at \mathbf{v}_{out}^2 and with Dirichlet conditions at \mathbf{v}_{in}^1 and \mathbf{v}_{out}^2 .

It is clear from (1.1) that

$$\Phi_{NN}(\lambda) = \Phi_{NN}^1(\lambda)\Phi_{DN}^2(\lambda) + \Phi_{ND}^1(\lambda)\Phi_{NN}^2(\lambda), \quad (3.1)$$

$$\Phi_{ND}(\lambda) = \Phi_{NN}^1(\lambda)\Phi_{DD}^2(\lambda) + \Phi_{ND}^1(\lambda)\Phi_{ND}^2(\lambda), \quad (3.2)$$

$$\Phi_{DN}(\lambda) = \Phi_{DN}^1(\lambda)\Phi_{DD}^2(\lambda) + \Phi_{DD}^1(\lambda)\Phi_{NN}^2(\lambda), \quad (3.3)$$

$$\Phi_{DD}(\lambda) = \Phi_{DN}^1(\lambda)\Phi_{DD}^2(\lambda) + \Phi_{DD}^1(\lambda)\Phi_{ND}^2(\lambda). \quad (3.4)$$

Using these identities we obtain an analogue of the Lagrange identity:

$$\begin{aligned} & \Phi_{ND}(\lambda)\Phi_{DN}(\lambda) - \Phi_{NN}(\lambda)\Phi_{DD}(\lambda) = \\ & (\Phi_{ND}^1(\lambda)\Phi_{DN}^1(\lambda) - \Phi_{NN}^1(\lambda)\Phi_{DD}^1(\lambda))(\Phi_{ND}^2(\lambda)\Phi_{DN}^2(\lambda) - \Phi_{NN}^2(\lambda)\Phi_{DD}^2(\lambda)). \end{aligned} \quad (3.5)$$

4. Auxiliary results

Denote by $s_{1,j}(\lambda, x_{1,j})$ the solution of the equation

$$-y_{1,j}'' + q_{1,j}(x_{1,j})y_{1,j} = \lambda^2 y_{1,j}, \quad j = 1, 2$$

on the edge of G_j incident with \mathbf{v}_{in}^j which satisfies $s_{1,j}(\lambda, 0) = s'_{1,j}(\lambda, 0) - 1 = 0$ and by $c_{1,j}(\lambda, x_{1,j})$ the solution which satisfies $c_{1,j}(\lambda, 0) - 1 = c'_{1,j}(\lambda, 0) = 0$.

Let us consider a boundary value problem on G_1 which consists of equations

$$-y''_{i,1} + q_{i,1}(x_{i,1})y_{i,1} = \lambda^2 y_{i,1}, \quad i = 1, 2, \dots, g_1, \quad (4.1)$$

continuity and Kirchhoff conditions at all interior vertices, conditions $y_{1,1}(0) = y'_{1,1}(0) - 1 = 0$ at \mathbf{v}_{in}^1 and no condition at \mathbf{v}_{out}^1 . There exists a solution of this problem, maybe not unique, of the form $Y_1(\lambda, \vec{x}_1) = (s_{1,1}(\lambda, x_{1,1}), y_{2,1}(\lambda, x_{2,1}), y_{3,1}(\lambda, x_{3,1}), \dots, y_{g_1,1}(\lambda, x_{g_1,1}))^T$, where $\vec{x}_1 = \{x_{1,1}, x_{2,1}, \dots, x_{g_1,1}\}^T$ is the coordinate vector corresponding to G_1 . In the same way we define the solution of the problem generated by equations (4.1), conditions $y_{1,1}(0) - 1 = y'_{1,1}(0) = 0$ at \mathbf{v}_{in}^1 and no condition at \mathbf{v}_{out}^1 : $U_1(\lambda, \vec{x}_1) = (c_{1,1}(\lambda, x_{1,1}), u_{2,1}(\lambda, x_{2,1}), u_{3,1}(\lambda, x_{3,1}), \dots, u_{g_1,1}(\lambda, x_{g_1,1}))^T$.

If we substitute $Y_1(\vec{x}_1)$ into equations (2.10)–(2.16) written for G_1 , we obtain

$$y_{g_1,1}(\lambda, x_{g_1,1}) = \frac{\Delta_{B_{g_1-1,1}}^D(\lambda)}{\Delta_1(\lambda)} c_{g_1,1}(\lambda, x_{g_1,1}) + \sum_{k=g-d(w)+1}^{g-1} \frac{\Delta_{A_{k,1}}^D(\lambda)}{\Delta_1(\lambda)} s_{g_1,1}(\lambda, x_{g_1,1}). \quad (4.2)$$

In the same way

$$u_{g_1,1}(\lambda, x_{g_1,1}) = \frac{\Delta_{B_{g_1-1,1}}^N(\lambda)}{\Delta_1(\lambda)} c_{g_1,1}(\lambda, x_{g_1,1}) + \sum_{k=g-d(w)+1}^{g-1} \frac{\Delta_{A_{k,1}}^N(\lambda)}{\Delta_1(\lambda)} s_{g_1,1}(\lambda, x_{g_1,1}). \quad (4.3)$$

In the same way we define the corresponding solutions for G_2 : $Y_2(\lambda, \vec{x}_2) = (s_{1,2}(\lambda, x_{1,2}), y_{2,2}(\lambda, x_{2,2}), y_{3,2}(\lambda, x_{3,2}), \dots, y_{g_2,2}(\lambda, x_{g_2,2}))^T$ which satisfies equations

$$-y''_{i,2} + q_{i,2}(x_{i,2})y_{i,2} = \lambda^2 y_{i,2}, \quad i = 1, 2, \dots, g_2, \quad (4.4)$$

and conditions $y_{1,2}(0) = y'_{1,2}(0) - 1 = 0$ at \mathbf{v}_{in}^2 and no condition at \mathbf{v}_{out}^2 and $U_2(\lambda, \vec{x}_2) = (c_{1,2}(\lambda, x_{1,2}), u_{2,2}(\lambda, x_{2,2}), u_{3,2}(\lambda, x_{3,2}), \dots, u_{g_2,2}(\lambda, x_{g_2,2}))^T$ which satisfies $u_{1,2}(0) - 1 = u'_{1,2}(0) = 0$ at \mathbf{v}_{in}^2 and no condition at \mathbf{v}_{out}^2 . Then

$$y_{g_2,2}(\lambda, x_{g_2,2}) = \frac{\Delta_{B_{g_2-1,2}}^D(\lambda)}{\Delta_2(\lambda)} c_{g_2,2}(\lambda, x_{g_2,2}) + \sum_{k=g-d(w)+1}^{g-1} \frac{\Delta_{A_{k,2}}^D(\lambda)}{\Delta_2(\lambda)} s_{g_2,2}(\lambda, x_{g_2,2}), \quad (4.5)$$

$$u_{g_2,2}(\lambda, x_{g_2,2}) = \frac{\Delta_{B_{g_2-1,2}}^N(\lambda)}{\Delta_2(\lambda)} c_{g_2,2}(\lambda, x_{g_2,2}) + \sum_{k=g-d(w)+1}^{g-1} \frac{\Delta_{A_{k,2}}^N(\lambda)}{\Delta_2(\lambda)} s_{g_2,2}(\lambda, x_{g_2,2}). \quad (4.6)$$

5. Parallel connection

Now we consider parallel connection of G_1 with G_2 , i.e. a graph G obtained by connection \mathbf{v}_{in}^1 with \mathbf{v}_{in}^2 and \mathbf{v}_{out}^1 with \mathbf{v}_{out}^2 . To simplify situation let us assume that \mathbf{v}_{in}^j and \mathbf{v}_{out}^j are pendant vertices of G_j for $j = 1, 2$, respectively, (see Fig. 2).

Let us denote by $\Phi_{NN}(\lambda)$, $\Phi_{ND}(\lambda)$, $\Phi_{DN}(\lambda)$ and $\Phi_{DD}(\lambda)$ the characteristic functions of the problems on G with Neumann conditions at $\mathbf{v}_{in}^1 = \mathbf{v}_{in}^2$ and $\mathbf{v}_{out}^1 = \mathbf{v}_{out}^2$, with Neumann conditions at $\mathbf{v}_{in}^1 = \mathbf{v}_{in}^2$ and Dirichlet at $\mathbf{v}_{out}^1 = \mathbf{v}_{out}^2$, with Dirichlet condition at $\mathbf{v}_{in}^1 = \mathbf{v}_{in}^2$ and Neumann at $\mathbf{v}_{out}^1 = \mathbf{v}_{out}^2$ and with Dirichlet conditions at $\mathbf{v}_{in}^1 = \mathbf{v}_{in}^2$ and at $\mathbf{v}_{out}^1 = \mathbf{v}_{out}^2$, respectively.

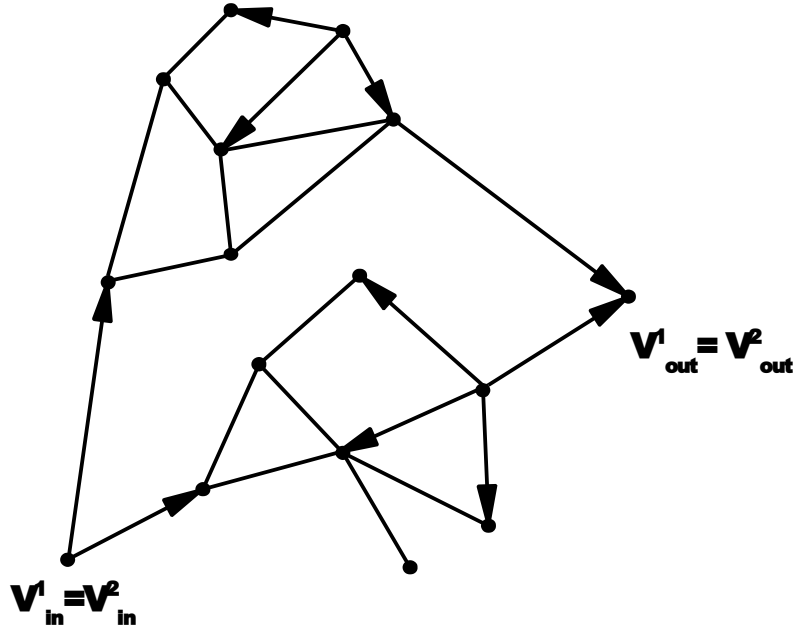


Fig.2

It is clear that

$$\Phi_{DD}(\lambda) = \Phi_{DD}^1(\lambda)\Phi_{DD}^2(\lambda), \quad (5.1)$$

$$\Phi_{DN}(\lambda) = \Phi_{DN}^1(\lambda)\Phi_{DD}^2(\lambda) + \Phi_{DD}^1(\lambda)\Phi_{DN}^2(\lambda), \quad (5.2)$$

$$\Phi_{ND}(\lambda) = \Phi_{ND}^1(\lambda)\Phi_{DD}^2(\lambda) + \Phi_{DD}^1(\lambda)\Phi_{ND}^2(\lambda). \quad (5.3)$$

Our aim is to express $\Phi_{NN}(\lambda)$ via $\Phi_{NN}^j(\lambda)$, $\Phi_{ND}^j(\lambda)$, $\Phi_{DN}^j(\lambda)$ and $\Phi_{DD}^j(\lambda)$ ($j = 1, 2$).

Let us look for solution of Neumann-Neumann problem on G in the form $\{R_{11}Y_1 + R_{21}U_1, R_{12}Y_2 + R_{22}U_2\}$, where R_{ij} are constants. Then the continuity condition at $\mathbf{v}_{in}^1 = \mathbf{v}_{in}^2$ is

$$R_{21} = R_{22}. \quad (5.4)$$

The Kirchhoff condition at $\mathbf{v}_{in}^1 = \mathbf{v}_{in}^2$ is

$$R_{11} = -R_{12}. \quad (5.5)$$

Continuity condition at $\mathbf{v}_{out}^1 = \mathbf{v}_{out}^2$ is

$$R_{11}y_{g_1,1}(\lambda, l_{g_1,1}) + R_{21}u_{g_1,1}(\lambda, l_{g_1,1}) = R_{12}y_{g_2,2}(\lambda, l_{g_2,2}) + R_{22}u_{g_2,2}(\lambda, l_{g_2,2}). \quad (5.6)$$

The Kirchhoff condition at $\mathbf{v}_{out}^1 = \mathbf{v}_{out}^2$ is

$$R_{11}y'_{g_1,1}(\lambda, l_{g_1,1}) + R_{21}u'_{g_1,1}(\lambda, l_{g_1,1}) + R_{12}y'_{g_2,2}(\lambda, l_{g_2,2}) + R_{22}u'_{g_2,2}(\lambda, l_{g_2,2}) = 0. \quad (5.7)$$

It is clear that if $y_{g_1,1}(\lambda, l_{g_1,1}) = 0$, then Y_1 is an eigenvector of the problem on G_1 with Dirichlet conditions at \mathbf{v}_{in}^1 and at \mathbf{v}_{out}^1 . It means that the set of zeros of $y_{g_1,1}(\lambda, l_{g_1,1})$ coincides with the set of zeros of $\Phi_{DD}^1(\lambda)$.

Using (2.29), (2.30) we obtain from (4.2), (4.3), (4.5) and (4.6) that $y_{g_1,1}(\lambda, l_{g_1,1}) = \frac{\Phi_{DD}^1(\lambda)}{\Delta_1(\lambda)}$, $u_{g_1,1}(\lambda, l_{g_1,1}) = \frac{\Phi_{ND}^1(\lambda)}{\Delta_1(\lambda)}$, $y_{g_2,2}(\lambda, l_{g_2,2}) = \frac{\Phi_{DD}^2(\lambda)}{\Delta_2(\lambda)}$, $u_{g_2,2}(\lambda, l_{g_2,2}) = \frac{\Phi_{ND}^2(\lambda)}{\Delta_2(\lambda)}$, $y'_{g_1,1}(\lambda, l_{g_1,1}) = \frac{\Phi_{ND}^1(\lambda)}{\Delta_1(\lambda)}$, $u'_{g_1,1}(\lambda, l_{g_1,1}) = \frac{\Phi_{NN}^1(\lambda)}{\Delta_1(\lambda)}$, $y'_{g_2,2}(\lambda, l_{g_2,2}) = \frac{\Phi_{ND}^2(\lambda)}{\Delta_2(\lambda)}$ and $u'_{g_2,2}(\lambda, l_{g_2,2}) = \frac{\Phi_{NN}^2(\lambda)}{\Delta_2(\lambda)}$.

The determinant of system (5.4)–(5.7) is

$$D(\lambda) \stackrel{def}{=} \det \begin{vmatrix} 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ \frac{\Phi_{DD}^1(\lambda)}{\Delta_1(\lambda)} & -\frac{\Phi_{DD}^2(\lambda)}{\Delta_2(\lambda)} & \frac{\Phi_{ND}^1(\lambda)}{\Delta_1(\lambda)} & -\frac{\Phi_{ND}^2(\lambda)}{\Delta_2(\lambda)} \\ \frac{\Phi_{DN}^1(\lambda)}{\Delta_1(\lambda)} & \frac{\Phi_{DN}^2(\lambda)}{\Delta_2(\lambda)} & \frac{\Phi_{NN}^1(\lambda)}{\Delta_1(\lambda)} & \frac{\Phi_{NN}^2(\lambda)}{\Delta_2(\lambda)} \end{vmatrix} =$$

$$\left(\frac{\Phi_{DD}^1(\lambda)}{\Delta_1(\lambda)} + \frac{\Phi_{DD}^2(\lambda)}{\Delta_2(\lambda)} \right) \left(\frac{\Phi_{NN}^1(\lambda)}{\Delta_1(\lambda)} + \frac{\Phi_{NN}^2(\lambda)}{\Delta_2(\lambda)} \right) -$$

$$\left(\frac{\Phi_{DN}^1(\lambda)}{\Delta_1(\lambda)} - \frac{\Phi_{DN}^2(\lambda)}{\Delta_2(\lambda)} \right) \left(\frac{\Phi_{ND}^1(\lambda)}{\Delta_1(\lambda)} - \frac{\Phi_{ND}^2(\lambda)}{\Delta_2(\lambda)} \right). \quad (5.8)$$

It is clear that to obtain the characteristic function $\Phi_{NN}(\lambda)$ the determinant (5.5) must be multiplied by $(\Delta_1(\lambda)\Delta_2(\lambda))^p$ (with some $p \geq 1$) to be an entire function. Let us show that this p must be 1.

To prove it we notice that $\Phi_{DD}^j(\lambda)$ is an entire function of exponential type $L^{(j)} = \sum_{k=1}^{g_j} l_k^{(j)}$, where $l_k^{(j)}$ are the lengths of the edges of the subgraph G_j and g_j is their number.

The function $\Delta_j(\lambda)$ is an entire function of exponential type $\sum_{k=2}^{g_j-1} l_k^{(j)} = L^{(j)} - l_1^{(j)} - l_{g_j}^{(j)}$.

The graph G_j consists of two edges e_1 and e_{g_j} and the subgraph G_j^0 (see Fig. 2) which are series connected. Then for $e_1^{(j)}$ and $e_{g_j}^{(j)}$ we have

$$c_{1,j}(\lambda, l_1^{(j)})s'_{1,j}(\lambda, l_1^{(j)}) - c'_{1,j}(\lambda, l_1^{(j)})s_{1,j}(\lambda, l_1^{(j)}) = 1, \quad (5.9)$$

$$c_{g_j,j}(\lambda, l_{g_j}^{(j)})s'_{g_j,j}(\lambda, l_{g_j}^{(j)}) - c'_{g_j,j}(\lambda, l_{g_j}^{(j)})s_{g_j,j}(\lambda, l_{g_j}^{(j)}) = 1. \quad (5.10)$$

Applying the analogue of Lagrange identity for series connected subgraphs (3.5) and using (5.6) and (5.7) we obtain

$$\Phi_{ND}^{(j)}(\lambda)\Phi_{DN}^{(j)}(\lambda) - \Phi_{NN}^{(j)}(\lambda)\Phi_{DD}^{(j)}(\lambda) =$$

$$\Phi_{ND}^{j0}(\lambda)\Phi_{DN}^{j0}(\lambda) - \Phi_{NN}^{j0}(\lambda)\Phi_{DD}^{j0}(\lambda),$$

where quantities with the zero upper index correspond to G_j^0 . Thus $\Phi_{ND}^{(j)}(\lambda)\Phi_{DN}^{(j)}(\lambda) - \Phi_{NN}^{(j)}(\lambda)\Phi_{DD}^{(j)}(\lambda)$ is an entire function of exponential type $L^{(j)} - l_{g_j}^{(j)} - l_{n_j}^{(j)}$. Taking into account that $\Delta_j(-\lambda) = \Delta_j(\lambda)$ and $\Delta_j(\bar{\lambda}) = \overline{\Delta_j(\lambda)}$ we conclude that

$$\frac{\Phi_{ND}^{(j)}(\lambda)\Phi_{DN}^{(j)}(\lambda) - \Phi_{NN}^{(j)}(\lambda)\Phi_{DD}^{(j)}(\lambda)}{\Delta_j(\lambda)} \rightarrow C, \quad |Im \lambda| \rightarrow \infty \quad (5.11)$$

where C is a real constant. Therefore, using (5.5) we obtain

$$\Delta_1(\lambda)\Delta_2(\lambda)D(\lambda) = \quad (5.12)$$

$$\begin{aligned} & \frac{\Phi_{DD}^1(\lambda)\Phi_{NN}^1(\lambda) - \Phi_{ND}^1(\lambda)\Phi_{DN}^1(\lambda)}{\Delta_1(\lambda)}\Delta_2(\lambda) + \\ & \Phi_{NN}^1(\lambda)\Phi_{DD}^2(\lambda) + \Phi_{NN}^2(\lambda)\Phi_{DD}^1(\lambda) + \Phi_{ND}^1(\lambda)\Phi_{DN}^2(\lambda) + \Phi_{ND}^2(\lambda)\Phi_{DN}^1(\lambda) + \\ & \frac{\Phi_{DD}^2(\lambda)\Phi_{NN}^2(\lambda) - \Phi_{ND}^2(\lambda)\Phi_{DN}^2(\lambda)}{\Delta_2(\lambda)}\Delta_1(\lambda). \end{aligned}$$

Due to (5.11) and (5.12) we conclude that $\Delta_1(\lambda)\Delta_2(\lambda)D(\lambda)$ is a meromorphic function with the set of zeros which coincides with the set of zeros of the entire function $\Phi_{NN}(\lambda)$, i.e. the characteristic function of the Neumann-Neumann problem on the whole graph. To prove this we notice that

$$|\Phi_{NN}(\lambda)|_{|Im\lambda| \rightarrow \infty} = C|Im\lambda|^{-p_1-p_2-2+q_1+q_2}e^{|Im\lambda|^L}(1+o(1))$$

where $L = L^{(1)} + L^{(2)} = \sum_{k=1}^{g_1} l_k^{(1)} + \sum_{k=1}^{g_2} l_k^{(2)}$, p_j is the number of vertices in G_j , and the number of vertices in G is $p_1 + p_2 - 2$. Due to (5.11) and (5.13) we obtain

$$|\Delta_1(\lambda)\Delta_2(\lambda)D(\lambda)|_{|Im\lambda| \rightarrow \infty} = \tilde{C}|Im\lambda|^{-p_1-p_2-2+q_1+q_2}e^{|Im\lambda|^L}(1+o(1)).$$

We have proved the following theorem.

Theorem *For a graph consisting of t parallel connected subgraphs*

$$\Phi_{NN}(\lambda) = \Delta_1(\lambda)\Delta_2(\lambda)D(\lambda)$$

where $D(\lambda)$ is given by (5.8).

If we connect m parallel subgraphs we obtain

$$\Phi_{NN} = \begin{pmatrix} 0 & 0 & \dots & \dots & \dots & 0 & 1 & -1 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 & 0 & -1 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 & 0 & \dots & \dots & 0 & -1 \\ 1 & 1 & \dots & \dots & \dots & 1 & 0 & 0 & \dots & \dots & \dots & 0 \\ \Phi_{DD}^1 & -\Phi_{DD}^2 & 0 & \dots & \dots & 0 & \Phi_{ND}^1 & -\Phi_{ND}^2 & 0 & \dots & \dots & 0 \\ \Phi_{DD}^1 & 0 & -\Phi_{DDwo}^3 & 0 & \dots & 0 & \Phi_{ND}^1 & 0 & \Phi_{ND}^2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \Phi_{DD}^1 & 0 & \dots & \dots & \dots & \Phi_{DD}^m & \Phi_{ND}^1 & 0 & \dots & \dots & \dots & \Phi_{ND}^m \\ \Phi_{DN}^1 & \Phi_{DN}^2 & \dots & \dots & \dots & \Phi_{DN}^m & \Phi_{NN}^1 & \Phi_{NN}^2 & \dots & \dots & \dots & \Phi_{NN}^m \end{pmatrix} \prod_{j=1}^m \Delta_j. \quad (5.13)$$

Formulae (3.1)-(3.5) and (5.1)-(5.3), (5.13) remain true for problems generated by finite dimensional analogue of Sturm-Liouville equation, so-called Stieltjes string equation.

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